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Cores of games with positive externalities

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Abstract

This paper introduces a core concept, called the γ-core, in the primitive framework of a strategic game. For a certain class of strategic games, it is a weaker concept than the strong Nash equilibrium, but in general stronger than the conventional α- and β- cores. We argue that the coalition formation process is an infinitely repeated game and show that the grand coalition forms if the γ-core is nonempty. This is a weaker sufficient condition than the previous such condition (Maskin (2003, Theorem 4)). As an application of this result, it is shown that the γ- core of an oligopolistic market is nonempty and thus the grand coalition forms.

Keywords: positive externalities, strategic game, core, repeated game, coalition formation.

JEL Classification: C7, D62

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1. Introduction

There are many economic environments in which the payoff of a coalition (i.e. a group of agents who decide to act together as one unit, relative to the rest of the agents) depends on how the agents *not* in the coalition cooperate. For example, in an oligopolistic market, the profit of a cartel is higher if the rest of the firms also form a cartel resulting in a duopoly than if they remain separate as profit maximizing oligopolists. Similarly, a free-rider's benefit depends on the level of cooperation among the other agents in the production of a public good. In such cases, what a coalition can achieve depends on the entire coalition structure and not just on the coalition in question. Thus, there can be many definitions of the characteristic function and therefore of the core as the payoff of the deviating coalition depends on what coalitions form in the complement after the deviation. Suppose N denotes the grand coalition of n agents. Then, we obtain one definition of core if each deviating coalition S presumes the resulting coalition structure after the deviation to be $\{S, N\backslash S\}$ and a different one if it presumes it to be instead $\{S, (T_k)_{k=1}^{K}\}$ $(1 \leq K < n)$ where $T_i \cap T_j = \emptyset$ for each $1 \leq i < j \leq K$ and $\cup_{k=1}^{K} T_k = N\backslash S$. The question thus arises which definition of core we should use for games with positive externalities.

This paper introduces a core concept, called the γ-core, in the primitive framework of a strategic game.[1] In this concept, each deviating coalition S presumes the resulting coalition structure after the deviation to be $\left\{S, \{i\}_{i\in N\backslash S}\right\}$, i.e., the rest of the coalition structure consists of all singletons. We argue below that this is a theoretically more compelling core concept for games with positive externalities.

The standard approach in the traditional cooperative game theory is to convert the strategic game with transferable utility into a characteristic function form game, and analyze the core of the cooperative game so induced. We will do the same except that the conversion is not standard. More specifically, the *worth* of coalition S is now defined as equal to its payoff in the Nash equilibrium between S and the other players acting individually, in which the members of S play

[1] This concept was originally introduced in Chander and Tulkens (1995, 1997) and Chander (2007), but in the context of a specific model of pollution. Presently, a general formulation in the framework of a strategic game is introduced.

their *joint* best reply strategy against the *individually* best reply strategies of the remaining players.

Using a strategic game as the primitive framework for defining the core is natural when there are widespread externalities across players and therefore across coalitions. However, this is not the only reason. Doing so also leads to additional interpretations and an even more general formulation of the concept.[2] In particular, it makes possible to define the γ-core both when the players can write binding agreements, as implicit in the definition of the strong Nash equilibrium, and when they cannot, as in the case of the coalition-proof Nash equilibrium. It also makes possible to interpret the γ-core both in terms of cooperative as well as non-cooperative game theory. In a certain class of games, if a game has a *unique* strong Nash equilibrium, then the γ-core is nonempty and consists of the *unique* imputation in which the payoffs of the players are equal to their strong Nash equilibrium payoffs. However, the γ-core may exist even if the game has no strong Nash equilibrium. Thus, the γ-core is a weaker concept than that of strong Nash equilibrium at least in a certain class of games. On the other hand, the γ-core is a stronger concept than the conventional α- and β- cores,[3] as we show that *in general* $w^{\alpha}(S) \leq w^{\beta}(S) \leq w^{\gamma}(S)$ for all $S \subset N$, where w^{α}, w^{β}, and w^{γ} are the respective characteristic functions.

Though the γ-core concept is quite appealing and is of an independent interest, a more important theoretical question is why this and not some other core concept (from among the several that are possible) is more compelling for games with positive externalities. The coalition theory is largely devoted to understanding how a group of agents may share the benefit of forming a coalition.[4] However, there is another more recent strand of work that aims at understanding whether agents have incentives to form a coalition, i.e., whether they *actually* decide to form a coalition. In the presence of externalities a player can derive benefits from the activities of a coalition without joining it, i.e., free ride. Therefore, the grand coalition may not form. Several non-cooperative coalition formation games have been proposed in recent years, see

[2] Alternatively, as will become apparent later, the γ-characteristic function can be defined through the partition function (due to Thrall and Lucas (1963)).
[3] These have been studied in various externalities contexts by Scarf (1971) and Zhao (1999) and in public goods context by Foley (1970), Moulin (1987), and Chander (1993) among others.
[4] See de Clippel and Serrano (2008) for a beautiful extension of the theory of Shapley value to games with externalities.

Greenberg and Weber (1993), Bloch (1996), Ray and Vohra (1999), and Yi (1997) among others. These studies employ a variety of rules of coalition formation and distribution of payoffs among coalition members. Though it is difficult to assess how the differences in the rules affect the equilibrium outcomes, they all lead to the conclusion that partitions finer than the grand coalition may form. Therefore, they can all be interpreted as providing sufficient conditions for the grand coalition not to form.

In a seminal paper, Maskin (2003) proposes a sufficient condition for the grand coalition to form (Maskin (2003, Theorem 4)).[5] More specifically, he shows that the grand coalition must form if the "core" exists. However, Maskin uses a core concept that is stronger than the γ-core in games with positive externalities. In his definition, the worth of a coalition S is determined by assuming that it faces the complementary coalition $N\backslash S$ which implies a higher worth for each coalition when externalities are positive. Restricting attention to three-player settings, Maskin considers a specific class of sequential bargaining procedures wherein he assumes that players can commit to refrain from forming coalitions with other players. However, he notes that if the players cannot make such commitments, the game develops into a war of attrition in which each player waits for the other two to form a coalition in the hope of free riding on them.

In this paper, we argue that if the γ-core is nonempty then it is indeed not credible for the players to commit to refrain from forming coalitions with other players. Thus, *each* player may wait for the other two to form a coalition. This leads us to formulate the coalition formation process as an infinitely repeated game in which the players decide strategically in each period whether to form a coalition or not. The game is repeated if no coalitions are formed. We show that if the γ-core is nonempty then the equilibrium strategy of the repeated game requires that if an individual player leaves, then as in the γ-characteristic function, the other two players do not cooperate. Hence, in equilibrium the grand coalition must form. Since the γ-core is a weaker concept, this implies a weaker sufficient condition for the grand coalition to form in games with positive externalities.

[5] Maskin (2003) introduces a comprehensive approach that covers both strategic and normative aspects of coalition formation.

Coalition formation in an oligopolistic market is perhaps one of the oldest problems in game theory.[6] As an application of our result, we show that the γ-core of an oligopolistic market exists and therefore the industry will become a monopoly unless prevented by law.

The contents of this paper are as follows. Section 2 introduces the γ-core concept in the framework of a strategic game and compares it with other related concepts. Section 3 introduces the infinitely repeated game of coalition formation and shows that the grand coalition forms if the γ-core is nonempty. Section 4 shows that the γ-core of the oligopoly game is nonempty. Section 5 draws the conclusion.

2. The general set-up

A strategic game with transferable utility is a triple (N,T,u) where $N=\{1,2,\ldots,n\}$ is the set of players, T_i is the set of strategies of player i, $T=T_1\times T_2\times\cdots\times T_n$ is the set of joint strategies, and $u_i, i\in N,$ is the utility or payoff of player i. A strategy profile of the players is $t=(t_1,\ldots,t_n)$. Let $t_{-i}\equiv(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)$ and $(t_i,t_{-i})\equiv(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n)$. A subset of players or a coalition is denoted by S and its complement by $N\setminus S$. It will be convenient to denote a strategy of a coalition S by $t_S\equiv(t_i)_{i\in S}$ and the set of all strategies of S by $T_S\equiv\times_{i\in S}T_i$. Let $t_{-S}\equiv(t_j)_{j\in N\setminus S}$ denote the strategies of the players not in S and $(t_S,t_{-S})\equiv(t_1,\ldots,t_n)$. A strong Nash equilibrium of the strategic game (N,T,u) is a strategy profile $\bar{t}\in T$ such that $\sum_{i\in S}u_i(\bar{t})\geq\sum_{i\in S}u_i(t_S,\bar{t}_{-S})$ for all $t_S\in T_S$ and all $S\subset N$.

2.1 A *Nash equilibrium relative to a coalition*

Definition 1 A strategy $\tilde{t}\in T$ is a Nash equilibrium relative to a coalition $S\subset N$ if $\sum_{i\in S}u_i(\tilde{t})\geq\sum_{i\in S}u_i(t_S,\tilde{t}_{-S})$ for all $t_S\in T_S$, and $u_j(\tilde{t})\geq u_j(t_j,\tilde{t}_{-j})$ for all $t_j\in T_j$ and $j\in N\setminus S$.

[6] d'Aspremont et al. (1983) were the first to study coalition formation in an oligopolistic market.

A Nash equilibrium is a Nash equilibrium relative to each singleton coalition, and a strong Nash equilibrium is a Nash equilibrium relative to each coalition $S \subset N$. On the other hand, if a strategy $\bar{t}$ is a Nash equilibrium relative to each coalition, then $\bar{t}$ is a strong Nash equilibrium. However, a game may have a Nash equilibrium relative to each coalition, but no strong Nash equilibrium. This is because a Nash equilibrium relative to a given coalition may not be a Nash equilibrium relative to other coalitions. We summarize these observations in the following proposition.

***Proposition* 1** A strategy $\bar{t} \in T$ is a strong Nash equilibrium of the game (N, T, u) if and only if it is a Nash equilibrium relative to each coalition $S \subset N$.

Both in a strong Nash equilibrium and in a Nash equilibrium relative to a coalition, the coalition is allowed complete freedom in choosing its strategy. We can define an alternative concept by requiring the strategy $\tilde{t}$ in Definition 1 to be immune to deviations by subcoalitions of coalition S in the same self-enforcing manner as in a coalition-proof Nash equilibrium (Bernheim, Peleg, and Whinston (1987a) and Moreno and Wooders (1996)) vis-à-vis a strong Nash equilibrium. This means that the strategy $\tilde{t}$ in Definition 1 must be such that $\tilde{t}_S$ is self-enforcing in the component game with strategies of the complement fixed at $\tilde{t}_{-S}$ and $\sum_{i \in S} u_i(\tilde{t}) \geq \sum_{i \in S} u_i(t_S, \tilde{t}_{-S})$ for all $t_S \in T_S$ that are self-enforcing in the component game. Such an equilibrium is not necessarily a coalition-proof equilibrium of the original game and makes the concept of a Nash equilibrium relative to a coalition a fully non-cooperative concept. All the concepts introduced below can be redefined in terms of this alternative definition. However, since such equilibriums rarely exist, we do not pursue this alternative concept here and continue to assume instead that the coalition has complete freedom in choosing its joint strategy.

For the sake of completeness, we provide sufficient conditions for the existence of a Nash equilibrium relative to a coalition. These conditions, though strong, are satisfied in many economic applications, including an oligopolistic market.

***Proposition* 2** For each $S \subset N$, the strategic game (N, T, u) has a Nash equilibrium relative to S, if (i) each T_i is a compact convex set of a Euclidean space, and (ii) each $u_i(t)$ is continuous and concave in $(t_1, \dots, t_n)$.

Note that condition (ii) requires $u_i(t)$ to be concave in $(t_1, \dots, t_n)$ and not in t_i alone.

2.2 *The γ-core*

The Nash equilibrium relative to a coalition associates a payoff with the coalition. If this equilibrium is not unique, we can select the one that gives the highest payoff to the coalition. This is clearly possible in games with transferable utilities and compact strategy sets. If the payoffs are equal, any randomly selected equilibrium will do. In this way, we can associate a unique payoff with each coalition.

Definition 2 The γ-characteristic function of a strategic game (N, T, u) is the function defined as $w^\gamma(S) = \sum_{i \in S} u_i(\tilde{t}), S \subset N$, where $\tilde{t} \in T$ is the Nash equilibrium relative to coalition S. That is, $\tilde{t}$ is such that $\sum_{i \in S} u_i(\tilde{t}) \geq \sum_{i \in S} u_i\,(t_S, \tilde{t}_{-S})$ for all $t_S \in T_S$, and $u_j(\tilde{t}) \geq u_j(t_j, \tilde{t}_{-j})$ for each $t_j \in T_j$ and $j \in N \setminus S$.

The pair (N, w^γ) is a *characteristic function form* game representation of the strategic game(N, T, u). The γ-core of the strategic game (N, T, u) or equivalently the core of the characteristic function form game (N, w^γ) is the set of payoff vectors x satisfying
(i) $\sum_{i \in N} x_i = w^\gamma(N)$ and (ii) for each $S \subset N, \sum_{i \in S} x_i \geq w^\gamma(S)$.

In many applications, including an oligopolistic market (see e.g. Bernheim, Peleg, and Whinston (1987a)), the Nash equilibrium relative to a coalition is unique for each coalition. For such games the γ-core is a weaker concept than the strong Nash equilibrium. This is because if such a game has a unique strong Nash equilibrium, then the core of the game (N, w^γ) is nonempty and consists of the unique imputation with payoffs equal to the strong Nash equilibrium payoffs. However, if the game (N, T, u) has no strong Nash equilibrium, the core of the game (N, w^{γ}) may still be nonempty. Thus, the γ-core is a weaker concept than the strong Nash equilibrium for at least the class of games in which the Nash equilibrium relative to a coalition is unique for each coalition.

2.3 *The* α- and β- cores

Let us also compare the γ-core with the α- and β- cores that, as noted earlier, have been used for long in the traditional cooperative game theory. The first is based on the assumption that the players outside a coalition adopt those strategies that are least favorable to the coalition. Thus, the maximum payoff of a coalition $S \subset N$ is $w^{\alpha}(S) = \max_{t_S \in T_S} \min_{t_{N\setminus S} \in T_{N\setminus S}} \sum_{i \in S} u_i(t_S, t_{N\setminus S})$. In words, $w^{\alpha}(S)$ represents the highest payoff that coalition S can guarantee itself no matter what strategies are adopted by the players outside the coalition. In this concept, coalition S moves first and chooses a strategy that maximizes its payoff and takes into account that coalition $N\setminus S$ will move next and choose a strategy that minimizes its (i.e. S's) payoff. In the second concept, the maximum payoff of a coalition S is defined as $w^{\beta}(S) = \min_{t_{N\setminus S} \in T_{N\setminus S}} \max_{t_S \in T_S} \sum_{i \in S} u_i(t_S, t_{N\setminus S})$. In words, $w^{\beta}(S)$ represents the maximum payoff that coalition S can be held down to no matter what strategies are adopted by its members. In this concept, $N\setminus S$ moves first and chooses a strategy that minimizes S's payoff and takes into account that S will move next and choose a strategy that maximizes it's (i.e.S's) payoff.

In the α- and β- concepts, the players outside coalition S form a coalition and either coalition S moves first and coalition $N\setminus S$ next after seeing the strategies of S or $N\setminus S$ moves first and S next after seeing the strategies of $N\setminus S$. In the γ- concept, the players outside S do not form a coalition and remain separate as singletons. Furthermore, S and the outside players move simultaneously. It is like coalition S and the remaining individual players deciding to go their separate ways pursuing their own goals. If coalition S suffers any loss in its payoff due to the actions of players not in S, it is incidental and not the intention. Such behavior amounts to noncooperation, but not to war on S by the players in $N\setminus S$ as in the α- and β- concepts.

It is well-known that the α- and β- concepts imply large cores. In fact, they often imply cores that are "too large". The easiest way to see this is to consider a duopoly. Here the α- and β- cores consist of all allocations that maximize the profit of the duopoly and give each firm at least zero payoff. This is because each firm can be pushed by the other firm to the point where it is not possible for it to earn any profit. Because of these limitations of the α- and β-cores, the need for an alternative concept which is based on more plausible behavioral assumptions has been often expressed in the literature. In Harsanyi (1977) words: "… in a variable-sum game one would rather expect that each side would try to find a suitable compromise between trying to maximize the costs of a conflict to the other side and trying to minimize the costs of a conflict to itself – in other words, between trying to minimize the joint payoff of the opposing coalition and trying to maximize the joint payoff of their own coalition."

The γ-core seems to fulfill the need for such an alternative concept. We formally show that the γ- concept implies in general a smaller core.

Proposition 3 The α-, β-, and γ- characteristic functions satisfy $w^{\alpha}(S) \leq w^{\beta}(S) \leq w^{\gamma}(S)$ for all $S \subset N$.[7]

Proof: We only need to show that $w^{\beta}(S) \leq w^{\gamma}(S)$, since the inequality $w^{\alpha}(S) \leq w^{\beta}(S)$ is well-known. Let $(\tilde{t}_S, \tilde{t}_{-S})$ and $(\hat{t}_S, \hat{t}_{-S})$ be such that $w^{\gamma}(S) = \sum_{i \in S} u_i(\tilde{t}_S, \tilde{t}_{-S})$ and $w^{\beta}(S) = \sum_{i \in S} u_i(\hat{t}_S, \hat{t}_{-S})$. Let $t_S^*(t_{-S}) = \arg\max_{t_S \in T_S} \sum_{i \in S} u_i(t_S, t_{-S}), t_{-S} \in T_{-S}$. Then, by definition, $w^{\beta}(S) = \sum_{i \in S} u_i(t_S^*(\hat{t}_{-S}), \hat{t}_{-S}) \leq \sum_{i \in S} u_i(t_S^*(t_{-S}), t_{-S})$ for all $t_{-S} \in T_{-S}$. In particular, $\sum_{i \in S} u_i(t_S^*(\hat{t}_{-S}), \hat{t}_{-S}) \leq \sum_{i \in S} u_i(t_S^*(\tilde{t}_{-S}), \tilde{t}_{-S}) = w^{\gamma}(S)$. Therefore, $w^{\beta}(S) \leq w^{\gamma}(S)$. ■

Examples are easily constructed in which the inequality is strict (see e.g. Chander (2007)). Finally, note that the grand coalition is an efficient coalition structure, since the coalition of all players can choose at least the same strategies as the players in any coalition structure.

3. The coalition formation game

[7] This also implies a consistency property of the γ-core in the sense that the γ- core solutions are not inconsistent with the α- and β- core solutions.

The earliest attempt to generalize characteristic functions to the case of externalities among coalitions is the introduction of partition function form games by Thrall and Lucas (1963). A partition function defines the worth of a coalition for each possible coalition structure that may be formed by the rest of the players.[8] It does not rule out a priori any coalition structure and is a complete list of what payoffs a coalition can ever get. However, not all payoffs of a coalition may be relevant as the players in the complement may not have incentives to form all coalition structures.[9]

Maskin (2003) uses an example of a three player partition function to develop his ideas. It represents a simple free rider problem created by a public good that can be produced by each coalition of two players.[10]

Example 1 The set of agents is $N = \{a, b, c\}$, and the partition function is:

$v(N) = 24;$
$v(\{a, b\}; \{\{a, b\}, \{c\}\} = 12, v\big(\{a, c\};\ \{\{a, c\}, \{b\}\}\big) = 13, v\big(\{b, c\}; \{\{b, c\}, \{a\}\}\big) = 14,$
$v\big(\{i\}, \{\{i\}; \{j, k\}\}\big) = 9$ for all $i, j, k \in N$,
$v(\{i\}, \{\{i\}; \{j\}, \{k\}\}\big) = 0$ for all $i, j, k \in N$.

Maskin shows that if the ordering of the players is i, j, and k, then the coalition structure $\{\{i\}, \{j, k\}\}$ is a subgame perfect equilibrium outcome of a sequential bargaining game. Let us see why.

3.1 *The sequential bargaining game*

[8] A strategic game can be converted into a partition function form game by defining a Nash equilibrium across coalitions in the same way as in Section 2 except that now the complement may not consist of all singletons.
[9] See Wooders and Page (2008) for a discussion along these lines.
[10] This example also features in de Clippel and Serrano (2008). A similar example was first used by Ray and Vohra (1999, Example 1.2).

Consider, e.g., the natural order a, b, and c. There are two possibilities. If a does not form a coalition with b, then he has to compete with him to induce c to form a coalition with one of them. Player a may reason that if he lets c join b then his payoff will be 9. Therefore, he will be willing to pay c no more than 4 =13-9. But b is happy to pay (a little more than) 4 and form a coalition with c resulting in a payoff of 10 =14-4 for him which is more than the payoff of 9 that he would get if he were to let c join a instead. On the other hand, if a forms a coalition with b, then by similar reasoning he will have to pay 10 to b and 9 to c resulting in a payoff of only 5 =24-10-9 for him. Hence, the coalition structure $\{\{a\},\{b,c\}\}$ is a subgame perfect equilibrium outcome of the sequential bargaining game with payoffs of (9, 10, 4). As Maskin notes, this result depends on the implicit assumption that a's decision not to form a coalition with b is irreversible.[11] By his commitment not to merge with b, a is able to force b and c to form a coalition and thus give himself the opportunity to free-ride.[12] If c is not convinced of a's commitment, then he may not form a coalition with b in the hope that it will force a and b to form a coalition instead and he will get the free-riding payoff of 9. Thus, if players cannot commit to refrain from forming coalitions with other players, the game develops into a war of attrition in which each player waits for the other two to form a coalition in the hope of free–riding.

Such a war of attrition is clearly a repeated game without discounting as each player decides at each moment of time whether to cooperate or not and, by assumption, there is no cost of waiting.[13] A fixed finite number of repetitions are, however, arbitrary and no different from a single play game. This is because no player will then have incentive to cooperate until the very last and the game degenerates into a single play game. It is thus natural to model the coalition formation process as an infinitely repeated game.

3.2 *The infinitely repeated game*

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[11] If not, coalition $\{b,c\}$ can sign up player a and achieve payoffs, say (9.5, 10.25, 4.25), which are higher for everyone. But if a's decision is reversible, then c may refuse to form a coalition with b in the first place.

[12] Since each player is a first mover in some order of play, this actually amounts to assuming implicitly that *each* player can commit not to form a coalition with other players.

[13] This is true in many real life externality problems. For example, the negotiations on climate change or on world trade are repeated games, though not without cost of waiting.

For the sake of a clear exposition, we shall continue to focus on the case of three players and then briefly describe the generalization to more than three players. We consider games in partition function form.

The set of players is $N = \{1,2,3\}$. A set $P = \{S_1, \dots, S_m\}$ is a partition of N if $S_i \cap S_j = \emptyset$ for all $i, j \in N, i \neq j,$ and $\cup_{i=1}^{m} S_i = N$. For each partition P and each coalition $S_i \in P$ the partition function $v(;)$ associates a real number $v(S_i; P)$. We assume that a coalition can achieve at least as much as the sum of what its parts can, i.e., the partition function $v(;)$ is *superadditive*. More formally, for each partition $P = \{S_1, \dots, S_m\}$ and each $K \subset \{1, \dots, m\}$, $v(S; P^{'}) \geq \sum_{k \in K} v(S_k; P)$ where $S = \bigcup_{k \in K} S_k$ and $P^{'} = P \backslash \{(S_k)_{k \in K}\} \cup \{S\}$. In particular, $\sum_{i=1}^{m} v(S_i; P) \leq v(N; \{N\})$ for every partition $P = \{S_1, \dots, S_m\}$ of N, i.e., the grand coalition is an efficient coalition structure. The partition function $v(;)$ represents a game with *positive externalities* if $v(\{i\}; \{\{i\}, \{j, k\}\} \geq v(\{i\}; \{\{i\}, \{j\}, \{k\}\}), \{i, j, k\} = N$.

If the members of a coalition $\{i, j\}$ in the coalition structure $\{\{i, j\}, \{k\}\}$ decide to dissolve the coalition, i.e., to not give effect to the coalition, then that is equivalent to forming the coalition structure $\{\{i\}, \{j\}, \{k\}\}$ and the payoffs of the three players are $v(\{i\}; \{\{i\}, \{j\}, \{k\}\}), i, j, k \in N$. Note that only a non-singleton coalition can be dissolved. The act of dissolving a coalition captures the idea that a coalition can do whatever its members can do individually. For instance, if the members of coalition $\{b, c\}$ in Example 1 decide not to produce the public good, then that is equivalent to dissolving the coalition and forming the coalition structure $\{\{a\}, \{b\}, \{c\}\}$.

The γ-core of the partition function form game (N, v) is nonempty if there exists an allocation (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = v(N; \{N\})$, $x_i \geq v(\{i\}; \{\{i\}, \{j\}, \{k\}\})$ for each $i \in N$, and $x_i + x_j \geq v(\{i, j\}; \{\{i, j\}, \{k\}\})(\geq 0)$ for each $i, j, k \in N$. Note that this definition does not require $x_i \geq v(\{i\}; \{\{i\}, \{j, k\}\})$. [14]

Each play of our infinitely repeated game starts from the finest coalition structure $\{\{1\}, \{2\}, \{3\}\}$ as the status quo and has two stages. In Stage 1, each player decides (non-

[14] It is worth noting that the γ-core of the partition function form game in Example 1 is nonempty.

cooperatively) whether to cooperate (announce C) or to not cooperate (announce NC). In Stage 2, all those players who announce C in the first stage form a coalition and decide (cooperatively) whether to give effect to the coalition or to dissolve it. All those players who announce NC in Stage 1 form singleton coalitions.

Note that the finest coalition structure $\{\{1\},\{2\},\{3\}\}$ is an outcome of Stage 2 if either two or more players announce NC in Stage 1 or two or more players announce C at Stage 1, but decide to dissolve the coalition at Stage 2.[15] The game is repeated if the outcome of Stage 2 is the finest coalition structure as everyone stands to gain from it.

Depending on the outcome of Stage 2, the payoffs of the players are as follows: if the outcome is the grand coalition, then the payoff of player i is $x_i^*, i \in N$, such that $x_1^* + x_2^* + x_3^* = v(N,\{N\})$, if the outcome is a coalition $\{\{i,j\},\{k\}\}$, then the payoffs of players i, j, and k are y_{ij}^*, y_{ji}^*, and $v(\{k\};\{\{k\},\{i,j\}\})$, respectively, such that $y_{ij}^* + y_{ji}^* = v(\{i,j\};\{\{i,j\},\{k\}\})$, and if the outcome is the finest coalition structure $\{\{i\},\{j\},\{k\}\}$, then the payoffs are $v(\{i\};\{\{i\},\{j\},\{k\}\}), i,j,k \in N$.

The payoffs of the players are *according* to a γ-core allocation if (x_1^*, x_2^*, x_3^*) is a γ-core allocation and $y_{ij}^* \leq x_i^*$ for all $i,j \in N$. The latter inequalities are possible, since (x_1^*, x_2^*, x_3^*) is a γ-core allocation and therefore $x_i^* + x_j^* \geq v(\{i,j\};\{\{i,j\},\{k\}\}) = y_{ij}^* + y_{ji}^*$.

This completes the description of the repeated game. It is worth noting that allowing repetitions of the game may actually reduce players' incentives to form coalitions. This is because each player may think that any miscalculation about others forming a coalition in the current game period can be corrected in future periods.

Theorem 4 If the γ-core of the partition function form game (N,v) is nonempty and the payoffs of the players are according to a γ-core allocation (x_1^*, x_2^*, x_3^*), then the grand coalition N is the unique equilibrium outcome of the repeated game.

[15] As will be seen below, it may be sometimes *optimal* for the members of a coalition to dissolve the coalition.

Proof: In order to obtain a sharper proof, we restrict to the case in which $x_i^* + x_j^* > v(\{i,j\}; \{\{i,j\},\{k\}\})$ and $y_{ij}^* < x_i^*$ for all $i,j,k \in N$. It will be clear to the reader that the proof holds also if the inequalities are not strict. We show that in the repeated game

(i) each player can *credibly* commit to dissolve a coalition if it does not include all players,

(ii) the grand coalition is the unique equilibrium outcome and the payoffs of the players are the γ-core allocation (x_1^*, x_2^*, x_3^*),

(iii) if the players do not commit to dissolve coalitions that do not include all players, then the equilibrium payoff of each player i is lower than x_i^*.

It is convenient to first prove (ii) assuming (i) and then prove (i) and (iii). Let w_i be the value of the repeated game to player $i, i \in N$. Given the players commitments as in (i) and their responses to it, we derive a reduced form of the repeated game as follows.

Remember that each play of the repeated game has two stages. If in some period, all players announce C in Stage 1, then the outcome of Stage 2 is the grand coalition and the payoffs are (x_1^*, x_2^*, x_3^*). On the other hand, if some player announces NC in Stage 1, then as per players' commitments any non-singleton coalition is dissolved and the outcome of Stage 2 is the coalition structure $\{\{1\},\{2\},\{3\}\}$. Thus, the game is repeated next period and the payoff from that for each player i is w_i one period later. The payoff matrix of the so-defined reduced form of the repeated game is as below.

Player 3

	C		NC	
	Player 2		**Player 2**	
Player 1	C	NC	C	NC
C	x_1^*, x_2^*, x_3^*	w_1, w_2, w_3	w_1, w_2, w_3	w_1, w_2, w_3
NC	w_1, w_2, w_3	w_1, w_2, w_3	w_1, w_2, w_3	w_1, w_2, w_3

To find a solution of the reduced form game, let us consider first mixed strategy Nash equilibriums. Let p_1, p_2, p_3 be the probabilities assigned by the three players to the C strategy. Then in equilibrium each player, say 1, should be indifferent between strategies NC and C. Thus, $w_1 = p_2p_3w_1 + (1 - p_2p_3)w_1 = p_2p_3x_1^* + (1 - p_2p_3)w_1$. If $x_1^* \geq w_1$, then C is the dominant strategy and the resulting payoff is $w_1 = x_1^*$, confirming the requirement for dominance.[16] Thus, C is a dominant strategy of each player and the equilibrium payoffs are $w_i = x_i^*, i \in N$.

We now prove (i). Suppose in Stage 1 of the two-stage game in some period of the repeated game, two players, say 2 and 3, announce C, but player 1 announces NC. Suppose further that in Stage 2, players 2 and 3 form coalition {2,3} and do not dissolve it. Such a deviation from their commitment would lead to payoffs of $y_{23}^* < x_2^*$ and $y_{32}^* < x_3^*$ for 2 and 3, respectively. However, if they adhere to their commitment and dissolve the coalition, the game will be repeated next period. Their payoffs from that will be x_2^* and x_3^*, which are higher than y_{23}^* and y_{32}^*,

[16] We can get strict dominance by assuming that the players value the same payoff a bit less if it occurs one period later. The equalities then become $w_1 = p_2p_3w_1' + (1 - p_2p_3)w_1' = p_2p_3x_1^* + (1 - p_2p_3)w_i'$ where $w_1' < w_1$, and C is clearly strictly dominant. That it is dominant follows from lim $w_1' \to w_1$.

respectively. Thus, it is ex post optimal for players 2 and 3 to dissolve the coalition which player 1 *must* take into account when deciding his strategy at Stage 1.[17] This proves (i).

We now prove (iii). Let p_1, p_2, p_3 be the probabilities assigned by the players to the C strategy when they are not committed to dissolve coalitions that do not include all players. Then, in the equilibrium of the corresponding reduced form of the repeated game the players should be indifferent between strategies C and NC. Thus, for each player, say 1,

$$w_1 = p_2 p_3 x_1^* + p_2(1-p_3)y_{12}^* + p_3(1-p_2)y_{13}^* + (1-p_2)(1-p_3)w_1$$
$$w_1 = p_2 p_3 v\big(\{1\};\{\{1\},\{2,3\}\}\big) + [p_2(1-p_3) + p_3(1-p_2) + (1-p_2)(1-p_3)]w_1.$$

These equalities imply

$$w_1 = \frac{[p_2(1-p_3)y_{12}^* + p_3(1-p_2)y_{13}^*] - p_2 p_3[v\big(\{1\};\{\{1\},\{2,3\}\}\big) - x_1^*]}{p_2(1-p_3) + p_3(1-p_2)}$$

$$< x_1^* - \frac{p_2 p_3\big[v\big(\{1\};\{\{1\},\{2,3\}\}\big) - x_1^*\big]}{p_2(1-p_3) + p_3(1-p_2)}.$$

If $v\big(\{1\};\{\{1\},\{2,3\}\}\big) \geq x_1^*$, then clearly $w_1 < x_1^*$. On the other hand, if $v\big(\{1\};\{\{1\},\{2,3\}\}\big) < x_1^*$, then

(a) if $p_2(1-p_3)y_{12}^* + p_3(1-p_2)y_{13}^* \geq [p_2(1-p_3) + p_3(1-p_2)]w_1$,
C is a dominant strategy for player 1 and $w_1 = [p_2 p_3 x_1^* + p_2(1-p_3)y_{12}^* + p_3(1-p_2)y_{13}^*] + (1-p_2)(1-p_3)w_1 < [1-(1-p_2)(1-p_3)]x_1^* + (1-p_2)(1-$ $p3w1$ which implies $w1<x1*,$

[17] Note that the argument here is not that players 2 and 3 can force player 1 to join them by threatening to dissolve the coalition (and thereby deny him the opportunity to free ride), but rather that given their commitment as in (i) and the players' responses to it, such an action is ex post optimal for players 2 and 3, i.e., a subgame perfect equilibrium strategy of the reduced game. Hence this is a credible commitment that player 1 should take into account when choosing his strategy in Stage 1 of the two-stage game.

(b) if $p_2(1-p_3)y_{12}^* + p_3(1-p_2)y_{13}^* < [p_2(1-p_3) + p_3(1-p_2)]w_1$ and C is *not* a dominant strategy,[18] $w_1 = p_2p_3v\big(\{1\};\{\{1\},\{2,3\}\}\big) + [p_2(1-p_3) + p_3(1-p_2) + (1-p_2)(1-p_3)]w_1 < p_2p_3x_1^* + (1-p_2p_3)w_1$ which implies $w_1 < x_1^*$.

This proves (iii), since as shown, $w_i < x_i^*$ for each $i \in N$ if the players do not commit to dissolve coalitions that do not include all players. Thus, committing to dissolve any coalition that does not include all players is not only a credible, but also an optimal strategy and the players will themselves *choose* to commit to not form coalitions that do not include all players. Hence, the grand coalition is the unique equilibrium of the repeated game. ■

An intuitive explanation for this result is as follows: since the partition function is superadditive and the γ-core is nonempty, $x_i^* + x_j^* \geq v\big(\{i,j\};\{\{i,j\},\{k\}\}\big) \geq v\big(\{i\};\{\{i\},\{j\},\{k\}\}\big) + v\big(\{j\};\{\{i\},\{j\},\{k\}\}\big)$ for all $i,j,k \in N$. This means that though the players i and j can benefit by forming the coalition $\{i,j\}$, they can benefit *even more* if instead the grand coalition is formed. Therefore, it is ex post optimal for players i and j to not form a coalition without the third player as the game would be then repeated until the grand coalition is formed.

A generalization of this result to partition functions with $n > 3$ follows from the fact that if the γ-core is nonempty, then superadditivity implies that every coalition structure that does not consist of all singletons has *at least* one non-singleton coalition whose payoff is lower. More formally, let $S_1, \dots, S_K$ be the non-singleton coalitions in the coalition structure $P = \{S_1, \dots, S_m\}$. Let $S = \bigcup_{k=1}^{K} S_k$ and $P' = P \backslash \{S_1, \dots, S_K\} \cup \{S\}$. Then, $\sum_{k=1}^{K} v(S_k;P) \leq v(S;P') \leq$ *i∈Sxi∗ (=k=1Ki∈Skxi∗),* since *v(;)* is superadditive and the γ-core is nonempty. This implies $v(S_k;P) \leq \sum_{i \in S_k} x_i^*$ for at least some non-singleton coalition S_k of partition P. If the members of such a non-singleton coalition dissolve the coalition, then another coalition among the remaining non-singleton coalitions will have a lower payoff, and so on … . It is therefore ex post optimal for the members of *all* non-singleton coalitions to dissolve their coalitions so that the game is repeated until the grand coalition is formed.

[18] If C is still a dominant strategy, then as in (a) $w_1 < x_1^*$.

This argument also suggests that for games with non-empty γ-cores the assumption of superadditivity can be replaced by the weaker assumption that the grand coalition is an efficient coalition structure and every coalition structure that does not consist of all singletons has at least one non-singleton coalition whose payoff is lower.

4. The γ-core of an oligopolistic market

In this Section we demonstrate the applicability of Theorem 4 to the much studied model of an oligopolistic market in that it is shown that the γ-core of an oligopolistic market in nonempty. Such a result is of interest for a number of reasons. First, it implies that the oligopolists will act together as a monopolist unless prevented by law. Second, Zhao (1999) and Radner (2001) show that the α- and β- cores of an oligopolistic market are nonempty.[19] Since the γ-core, as shown, is in general smaller than the α- and β- cores, proving its existence is an alternative proof for the existence of the α- and β- cores. Third, as is well-known, the oligopoly game does not have a strong Nash equilibrium. Thus, the existence of γ-core illustrates the point made earlier that it is a weaker concept than the strong Nash equilibrium, at least for a certain class of games.

4.1 *The model*

The set of oligopolistic firms is $N = \{1, \dots, n\}$. Let $p(q)$ denote the inverse demand function faced by these firms, where q is the total demand. We assume that the inverse demand function is strictly decreasing, i.e., $p^{'}(q) < 0$. Let $c_i(q_i)$ denote the cost function of firm i. We assume that the cost function of each firm is increasing and strictly convex, i.e., $c_i^{'}(q_i) > 0$, and $c_i^{''}(q_i) > 0, q_i > 0$. Assuming strict convexity of the cost functions enables us to avoid the problem of multiple solutions, but it should be clear to the reader that the results below also hold if the cost function is linear. The profit function of each firm i is

$$\pi_i(q_1, \dots, q_n) = p(q)q_i - c_i(q_i),$$

[19] The α- and β- cores of an oligopolistic market actually coincide and, as noted in Section 2.3, are usually very large.

where $q = \sum_{j \in N} q_j$. In order to avoid corner solutions, we assume that there exists a q^0 such that $c_i'(q^0) > p(q^0)$ and $c_i'(0) = 0 < p(nq^0), i \in N$. This assumption implies that a profit maximizing firm will never produce an output larger than q^0 even if it has the capacity to do so and will always produce a positive amount.[20] Many quadratic cost functions and linear demand functions satisfy these assumptions. We assume further that the revenue function $p(q)q_i$ of each firm i is concave, in $q_1, \dots, q_n$. Thus, the marginal revenue $p(q) + p'(q)q_i$ of each firm i is non-increasing with total demand q, i.e., $p'(q) + p''(q)q_i \leq 0$ for each fixed $q_i \geq 0$. Note that this condition is satisfied, if the inverse demand function $p(q)$ is decreasing and concave.

4.2 *The oligopoly game*

Let $T_i = [0, q^0], T = T_1 \times \cdots \times T_n$, and $\pi = \big(\pi_1(.), \dots, \pi_n(.)\big)$. We shall refer to the strategic game (N, T, π) as the *oligopoly game*. Remember that the players are not assumed to be identical.

Lemma 5 There exists a unique Nash equilibrium$(\bar{q}_1, \dots, \bar{q}_n)$ of the oligopoly game (N, T, π).

Proof: Clearly, each $\pi_i(.)$ is concave in $q_1, \dots, q_n$. Proposition 3 therefore implies that there exists a Nash equilibrium, say $(\bar{q}_1, \dots, \bar{q}_n)$. Suppose contrary to the assertion that the game has another Nash equilibrium, say $(\bar{\bar{q}}_1, \dots, \bar{\bar{q}}_n)$, and $(\bar{q}_1, \dots, \bar{q}_n) \neq (\bar{\bar{q}}_1, \dots, \bar{\bar{q}}_n)$. Without loss of generality, let $\bar{q} = \sum_{i \in N} \bar{q}_i \geq \sum_{i \in N} \bar{\bar{q}}_i = \bar{\bar{q}}$. Since $(\bar{q}_1, \dots, \bar{q}_n) \neq (\bar{\bar{q}}_1, \dots, \bar{\bar{q}}_n)$, $\bar{q}_i > \bar{\bar{q}}_i$ for at least one i. Furthermore, $p'(\bar{\bar{q}})\bar{\bar{q}}_i + p(\bar{\bar{q}}) > p'(\bar{\bar{q}})\bar{q}_i + p(\bar{\bar{q}}) \geq p'(\bar{q})\bar{q}_i + p(\bar{q})$, since $\bar{q} \geq \bar{\bar{q}}$ and by assumption the marginal revenue of each firm is non-increasing with total demand. From the first order conditions for Nash equilibrium $c_i'(\bar{\bar{q}}_i) = p'(\bar{\bar{q}})\bar{\bar{q}}_i + p(\bar{\bar{q}}) > p'(\bar{q})\bar{q}_i + p(\bar{q}) = c_i'(\bar{q}_i)$ implying $\bar{q}_i < \bar{\bar{q}}_i$, which is a contradiction. ■

Let $S \subset N$ be some coalition and let $(q_1^S, \dots, q_n^S)$ denote the Nash equilibrium relative to S.

[20] The output level q^0 is not to be confused with the production capacity constraints that are often assumed in models of oligopolistic markets.

Proposition 6 For each coalition $S \subset N$, there exists a unique Nash equilibrium relative to S, say $(q_1^S, \dots, q_n^S)$, such that (i) $q^S = \sum_{i \in N} q_i^S \leq \sum_{i \in N} \bar{q}_i = \bar{q}$, and (ii) $q_j^S \geq \bar{q}_j$ for each $j \in N \backslash S$, i.e., if a cartel forms the total industry output is lower, but the output of each independent firm is higher.

Proof: By Proposition 2, the game (N, T, π) has a Nash equilibrium relative to each coalition $S \subset N$. By similar arguments as in Lemma 5, it is also unique. We prove the rest. (i) Suppose contrary to the assertion that $q^S > \bar{q}$. Then, either $q_j^S > \bar{q}_j$ for some $j \in N \backslash S$ or $\sum_{i \in S} q_i^S > \sum_{i \in S} \bar{q}_i$. In the former case, $c_j^{'}(\bar{q}_j) = p(\bar{q}) + \bar{q}_j p^{'}(\bar{q}) > p(\bar{q}) + q_j^S p^{'}(\bar{q}) \geq p(q^S) + q_j^S p^{'}(q^S) = c_j^{'}(q_j^S) \Longrightarrow \bar{q}_j > \bar{q}_j^S$, which is a contradiction. In the latter case, for each $i \in S$, $c_i^{'}(q_i^S) = \sum_{j \in S} q_j^S p^{'}(q^S) + p(q^S) < \sum_{i \in S} \bar{q}_i p^{'}(q^S) + p(q^S) \leq \sum_{i \in S} \bar{q}_i p^{'}(\bar{q}) + p(\bar{q}) \leq \bar{q}_i p^{'}(\bar{q}) + p(\bar{q}) = c_i^{'}(\bar{q}_i) \Longrightarrow \bar{q}_i > q_i^S$, which contradicts our supposition that $\sum_{i \in S} q_i^S > \sum_{i \in S} \bar{q}_i$. Therefore, $q^S \leq \bar{q}$.

(ii) Suppose contrary to the assertion that $q_j^S < \bar{q}_j$ for some $j \in N \backslash S$. Then, since $q^S \leq \bar{q}$ as shown, $c_j^{'}(\bar{q}_j) = p(\bar{q}) + \bar{q}_j p^{'}(\bar{q}) < p(\bar{q}) + q_j^S p^{'}(\bar{q}) \leq p(q^S) + q_j^S p^{'}(q^S) = c_j^{'}(q_j^S) \Longrightarrow \bar{q}_j < \bar{q}_j^S$, which contradicts our supposition that $q_j^S < \bar{q}_j$ for each $j \in N \backslash S$. Therefore, $q_j^S \geq \bar{q}_j$ for all $j \in N \backslash S$. ■

A standard method to prove the existence of core of a TU game (see e.g. Helm (2001)) comes from the well-known Bondareva-Shapley theorem, which specifies conditions that are both necessary and sufficient for a non-empty core. It uses the following concept of a balanced collection of coalitions.

Given the set of players N, let $\mathbb{C}$ denote the set of all coalitions that can be formed and for each $i \in N$, let $\mathbb{C}_i = \{S \in \mathbb{C}: i \in S\}$ denote the subset of all coalitions of which i is a member. Then, $\mathbb{C}$ is a balanced collection of coalitions, if for each coalition $S \in \mathbb{C}$, there exists a $\delta_S \in [0,1]$ such that $\sum_{S \in \mathbb{C}_i} \delta_S = 1$ for every $i \in N$.

Proposition (Bondareva (1963); Shapley (1967)) A characteristic function form game (N, w) has a nonempty core if and only if $\sum_{S \in \mathbb{C}} \delta_S w(S) \leq w(N)$ for every balanced collection of coalitions $\mathbb{C}$.

Proposition 6 implies that $\sum_{i \in N \setminus S} q_i^S \geq \sum_{i \in N \setminus S} \bar{q}_i$ and $\sum_{i \in S} q_i^S \leq \sum_{i \in S} \bar{q}_i$ for each $S \subset N$. This leads to an important inequality for any balanced collection of coalitions. Before establishing that inequality, however, we note a useful accounting identity: for each $i \in N$, $\sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \notin S} \bar{q}_j = \sum_{S \in \mathbb{C}_i} \delta_S (\sum_{j \in N} \bar{q}_j - \sum_{j \in S} \bar{q}_j) = \sum_{j \in N} \bar{q}_j - \sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \in S} \bar{q}_j = \sum_{S \in \mathbb{C}} \delta_S \sum_{i \in S} \bar{q}_i - \sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \in S} \bar{q}_j = \sum_{S \in \mathbb{C} \setminus \mathbb{C}_i} \delta_S \sum_{j \in S} \bar{q}_j$.

Lemma 7 For each $i \in N$, $\sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \in N} q_j^S \geq \sum_{i \in N} \sum_{S \in \mathbb{C}_i} \delta_S q_i^S$, i.e., the average total output of all firms is not smaller than the total average output of all firms, where the average is taken over the coalitions containing player i.

<u>Proof</u>: For any arbitrary i,

$$\begin{aligned}
\sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \in N} q_j^S &= \sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \in S} q_j^S + \sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \notin S} q_j^S \\
&\geq \sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \in S} q_j^S + \sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \notin S} \bar{q}_j \quad \text{(by Proposition 6)} \\
&= \sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \in S} q_j^S + \sum_{S \in \mathbb{C} \setminus \mathbb{C}_i} \delta_S \sum_{j \in S} \bar{q}_j \quad \text{(by the accounting identity)} \\
&\geq \sum_{S \in \mathbb{C}_i} \delta_S \sum_{j \in S} q_j^S + \sum_{S \in \mathbb{C} \setminus \mathbb{C}_i} \delta_S \sum_{j \in S} q_j^S \quad \text{(by Proposition6)} \\
&= \sum_{S \in C} \delta_S \sum_{j \in S} q_j^S = \sum_{i \in N} \sum_{S \in \mathbb{C}_i} \delta_S q_i^S. \quad \blacksquare
\end{aligned}$$

Let $w^\gamma(.)$ denote the γ-characteristic function of the oligopoly game (N, T, π), i.e., $w^\gamma(S) = \sum_{i \in S} \pi_i(q_1^S, \dots, q_n^S)$, $S \subset N$, where $(q_1^S, \dots, q_n^S)$ is the Nash equilibrium relative to S.

Theorem 8 The characteristic function form game (N, w^γ) has a nonempty core.

<u>Proof</u>: In view of the Bondareva-Shapley theorem, we need to show that $\sum_{S \in \mathbb{C}} \delta_S w^\gamma(S) \leq w^\gamma(N)$ for any balanced collection of coalitions $\mathbb{C}$. By definition

$$\sum_{S\in\mathbb{C}} \delta_S w^\gamma(S) = \sum_{S\in\mathbb{C}} \delta_S \sum_{i\in S} \pi_i\,(q_1^S, \dots, q_n^S) = \sum_{i\in N} \sum_{S\in\mathbb{C}_i} \delta_S\, \pi_i(q_1^S, \dots, q_n^S)$$

$$\leq \sum_{i\in N} \pi_i(\sum_{S\in\mathbb{C}_i} \delta_S(q_1^S, \dots, q_n^S)) \text{ (since } \sum_{S\in\mathbb{C}_i} \delta_S = 1 \text{ and each } \pi_i(.) \text{ is concave)}$$

$$= \sum_{i\in N}[p(\sum_{S\in\mathbb{C}_i} \delta_S \sum_{j\in N} q_j^S) \sum_{S\in\mathbb{C}_i} \delta_S q_i^S - c_i(\sum_{S\in\mathbb{C}_i} \delta_S q_i^S)]$$

$$\leq p\big(\sum_{i\in N} \sum_{S\in\mathbb{C}_i} \delta_S q_i^S\big) \sum_{i\in N} \sum_{S\in\mathbb{C}_i} \delta_S q_i^S - \sum_{i\in N} c_i(\sum_{S\in\mathbb{C}_i} \delta_S q_i^S) \quad \text{(by Lemma 7)}$$

$$\leq w^\gamma(N),$$

since $w^\gamma(N) \geq \sum_{i\in N}[p(q)q_i - c_i(q_i)]$ for all $(q_1, \dots, q_n)$ and $q = \sum_{i\in N} q_i$. ■

The γ-characteristic function of an oligopolistic market is in general not superadditive. But the grand coalition is an efficient coalition structure and in many cases including that of identical firms every partition of N that does not consist of all singletons has at least one non-singleton coalition whose payoff is lower.

5. Conclusion

This paper has introduced the γ-core concept in the primitive framework of a strategic game and presented an analysis involving both cooperative and non-cooperative ideas to argue that it is a compelling core concept for games with positive externalities. It was shown that in a general partition function form game, it is not only credible, but also optimal for the players to commit to not form coalitions that do not include all players. Therefore, only that payoff of a coalition is relevant that is obtained when the rest of the coalition structure consists of all singletons.

One natural question is whether that also applies to axiomatic analysis of games with externalities, i.e., whether a useful normative analysis such as the Shapley theory can be derived by similarly ignoring all but those payoffs of coalitions. In an important paper, de Clippel and Serrano (2008) extend the theory of Shapley value to games with externalities using a set of axioms similar to those behind the Shapley value in games without externalities. However, they assume that the grand coalition forms and ignore whether the players have incentives to actually form the grand coalition. This is a natural assumption in games without externalities, but not in games with externalities. As shown above, the grand coalition forms in games with positive externalities if the γ-core is nonempty. However, assuming that the γ-core is nonempty raises a consistency issue. This is because the γ-characteristic function, by definition, ignores all but those payoffs of coalitions that obtain when the rest of the coalition structure consists of all singletons. Consistency requires that any axiomatic analysis that begins with the assumption that the grand coalition forms should also ignore all but those payoffs? De Clippel and Serrano (2008, Proposition 3) confirm that doing so is indeed consistent with the axioms of anonymity, efficiency, and marginality when suitably extended to partition function form games. In other words, only the same payoffs of coalitions, as in the γ-characteristic function, should be taken into account in the determination of the Shapley value of a partition function form game. Thus, our strategic approach to coalition formation reinforces the normative approach in de Clippel and Serrano (2008).

The application of our analysis to an oligopolistic market with heterogeneous firms has shown that the firms have strong incentives to merge into a monopoly unless prevented by law. Other applications to consider in our future work include the standard model of a public good (see e.g. Chander (1993)). It is worth pointing out here that Theorem 4 is quite general in that it does not depend on the fact that the externalities are assumed to be positive. The same analysis can be easily extended to games with negative or mixed externalities.

Finally, the approach in this paper is in a sense similar to Aumann and Dreze (1974) and Shenoy (1979) in that we also assume that a coalition only gets its own worth. Thus, the total payoff of the coalitions in any coalition structure is less than the grand coalition payoff. On the other hand, Hart and Kurz (1983) assume that the grand coalition payoff is available for distribution among the coalitions whatever the coalition structure. In other words, the total payoff of the coalitions is same irrespective of what coalition structure forms.[21]

[21] An implication of this is that the players are indifferent between forming the grand coalition and remaining all separate.

References

1. d'Aspremont, C., J. J. Gabszewicz, A. Jacquemin, and J. A. Weymark (1983), "On the stability of collusive price leadership", *Canadian Journal of Economics* 16, 17-25.

2. Aumann, R. J. and J. Dreze (1974), "Cooperative games with coalition structures," *International Journal of Game Theory,* 3, 217-237.

3. Bernheim, B. D., B. Peleg, and D. Whinston (1987a), "Coalition-proof Nash equilibria: I. concepts", *Journal of Economic Theory,* 42, 1-12.

4. Bernheim, B. D., B. Peleg, and D. Whinston (1987b), "Coalition-proof Nash equilibria: II. Applications", *Journal of Economic Theory,* 42, 13-29.

5. Bloch, F. (1996), "Sequential formation of coalitions in games with externalities and fixed payoff division", *Games and Economic Behavior*, 14, 135-153.

6. Bondareva, O. N. (1963), "Some applications of linear programming methods to the theory of cooperative games" [in Russian]. Problemy Kibernetiki, 10, 119-139.

7. Chander, P. (1993), "Dynamic procedures and incentives in public good economies", *Econometrica*, 61, 1341-54.

8. Chander, P. (2007), "The gamma-core and coalition formation", *International Journal of Game Theory,* 2007: 539-556.

9. Chander, P. (2008), "Coalitional equilibria", paper presented at the Singapore Economic Theory Conference, held at National University of Singapore, August 11-12.

10. Chander, P. and H. Tulkens (1995), "A core-theoretic solution for the design of cooperative agreements on transfrontier pollution", *International Tax and Public Finance*, 2, 279-93.

11. Chander, P. and H. Tulkens (1997), "The core of an economy with multilateral environmental externalities", *International Journal of Game Theory,* 26, 379-401.

12. De Clippel, G. and R. Serrano (2008), "Marginal contributions and externalities in the value", *Econometrica,* 2008:1413-1436.

13. Foley, D. (1970), "Lindahl solution and the core of an economy with public goods", *Econometrica,* 38, 66-72.

14. Greenberg, J. and S. Weber (1993), "Stable coalition structures with a unidimensional set of alternatives", *Journal of Economic Theory,* 60, 62-82.

15. Harsanyi, J. C. (1977), *Rational Behavior and Bargaining Equilibrium in Games and Social Situations,* Cambridge University Press, Cambridge, Chapter 11.

16. Hart, S. and M. Kurz (1983), “Endogenous formation of coalitions”, *Econometrica*, 51, 1047-1064.

17. Helm, C. (2001), “On the existence of a cooperative solution for a coalitional game with externalities”, *International Journal of Game Theory*, 30, 141-147.

18. Maskin, E. (2003), “Bargaining, coalitions and externalities”, Presidential Address to the Econometric Society, Institute for Advanced Study, Princeton.

19. Moreno, D. and J. Wooders (1996), “Coalition-proof equilibrium”, *Games and Economic Behavior*, 17, 80-112.

20. Moulin, H. (1987), “Egalitarian-equivalent cost sharing of a public good”, *Econometrica*, 55, 963-76.

21. Radner, R. (2001), “On the core of a cartel”, in *Economic Essays*, G. Debreu, W. Neuefeind, and W. Trockel (eds.), Springer, Berlin, 315-331.

22. Ray, D. and R. Vohra (1999), “A theory of endogenous coalition structures”, *Games and Economic Behavior*, 26, 286-336.

23. Scarf, H. (1971), "On the existence of a cooperative solution for a general class of *N*-person games”, *Journal of Economic Theory*, 3, 169-81.

24. Shapley, L.S. (1967), “On balanced sets and cores”, *Naval Research Logistics Quarterly*, 14, 453-460.

25. Shenoy, P. P. (1979), “On coalition formation: a game theoretic approach”, *International Journal of Game Theory*, 8, 133-164.

26. Thrall, R. and W. Lucas (1963), “N-person games in partition function form”, *Naval Research Logistics Quarterly*, 10, 281-98.

27. Wooders, M. and F. H. Page, Jr. (2008), “Coalitions” in *The New Palgrave Dictionary of Economics*, 2nd edition, N. D. Steven and L. E. Blume (eds.), Palgrave Macmillan.

28. Zhao, J. (1999), “A β-core existence result and its application to oligopoly markets”, *Games and Economic Behavior*, 27, 153-168.

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